# THE PAUCITY PROBLEM FOR CERTAIN PAIRS OF DIAGONAL EQUATIONS

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## 1. Introduction

Associated with a given system of symmetric diagonal diophantine equations, the paucity problem is that of establishing that the number of non-diagonal solutions grows more slowly than the corresponding number of diagonal solutions. Let $k$ and $n$ be natural numbers with $k > n$, and consider, by way of example, the number $S_{k,n}(B)$ of integral solutions of the pair of equations

$$\sum_{i=1}^{3}(x_i^k - y_i^k) = \sum_{i=1}^{3}(x_i^n - y_i^n) = 0, \tag{1}$$

with $1 \leqslant x_i, y_i \leqslant B$ $(1 \leqslant i \leqslant 3)$. Let $T(B)$ denote the number of diagonal solutions counted by $S_{k,n}(B)$, that is, solutions in which $\{x_1, x_2, x_3\} = \{y_1, y_2, y_3\}$. Then, as a consequence of work of a number of authors (see [7, 9, 14, 15, 19, 20]), it is now known that whenever $k > 2$, there is a positive number $\delta$ for which the estimate

$$S_{k,n}(B) - T(B) \ll B^{3-\delta}$$

holds. On noting that $T(B) = 6B^3 + O(B^2)$, it is evident that there is a paucity of non-diagonal solutions in the class of systems (1). The proofs of such estimates follow two very different courses. In one direction, there is the sieve method of Greaves [6, 7], itself motivated by work of Hooley [10, 11, 12] on sums of two $k$-th powers. Alternatively, one has methods based on powerful estimates for the number of integral or rational points on curves and surfaces, and here it may be profitable to exploit creative slicing and use of polynomial identities (aside from the above cited work of Heath-Brown, Skinner, Tsui and Wooley, see also Bombieri and Pila [2] and Heath-Brown [8]).

The purpose of this note is to investigate a paucity problem for pairs of equations, now of the same degree, in rather more variables than are present in the aforementioned system. A noteworthy feature of our method is its use of the Hardy-Littlewood method. This is in marked contrast with the longstanding philosophy that this method should be ineffective for paucity problems.

When $k \geqslant 2$, let $U_k(B)$ denote the number of integral solutions of the pair of equations

$$x_1^k + x_2^k - y_1^k - y_2^k = x_3^k + x_4^k - y_3^k - y_4^k = x_5^k - y_5^k, \tag{2}$$

The second author was supported, in part, by NSF grant DMS-0140523. Both authors are grateful to Max Planck Institut für Mathematik in Bonn for generous hospitality during the period in which this paper was written.

with $1 \leqslant x_i, y_i \leqslant B$ $(1 \leqslant i \leqslant 5)$. Also, let $D(B)$ denote the number of solutions counted by $U_k(B)$ in which

$$\{x_1, x_2\} = \{y_1, y_2\}, \quad \{x_3, x_4\} = \{y_3, y_4\} \quad \text{and} \quad x_5 = y_5.$$

Of course $D(B)$ is indeed independent of $k$, and one has $D(B) = 4B^5 + O(B^4)$. In our first theorem, we establish the paucity of non-diagonal solutions in the system (2) when $k \geqslant 3$.

**Theorem 1.** *Suppose that $k \geqslant 3$. Then, for each $\varepsilon > 0$, one has*

$$U_k(B) - D(B) \ll B^5 (\log B)^{\varepsilon - 3}.$$

For comparison, the estimate $U_k(B) \ll B^{5+\varepsilon}$ was discovered only recently by the authors [4], and plays an important role in [5]. The condition that $k \geqslant 3$ is very relevant as $U_2(B)$ behaves quite differently. It would not be difficult to establish that $U_2(B) \sim CB^6$ for a suitable constant $C$.

For very large values of $k$, one may employ recent work of Heath-Brown [8, 9] in order to derive conclusions sharper than those presented in Theorem 1. We require some notation before we can formulate the result. When $1 \leqslant i \leqslant 4$ let $j = j(i)$ be defined by $\{i, j\} = \{2l - 1, 2l\}$ for either $l = 1$ or $l = 2$. For any solution of (2), we write $\widehat{x}_i = x_{j(i)}$ and $\widehat{y}_i = y_{j(i)}$. A solution of (2) is referred to as semi-diagonal if there are pairs $(r, s) \in \{1, 2\}^2$, $(t, u) \in \{3, 4\}^2$ such that

$$x_r = y_s, \quad x_t = y_u, \quad \widehat{x}_r = \widehat{x}_t = x_5, \quad \widehat{y}_s = \widehat{y}_u = y_5,$$

and the number of such solutions counted by $U_k(B)$ is denoted by $\Delta(B)$. Again, it is apparent that $\Delta(B)$ is independent of $k$, and one has $\Delta(B) = 16B^4 + O(B^3)$.

We also define $D_k^*(B)$ to be the number of solutions of the system (2) counted by $U_k(B)$ in which $x_5 = y_5$. Consequently, if $V_k(B)$ denotes the number of integral solutions of the equation

$$x_1^k + x_2^k = y_1^k + y_2^k,$$

with $1 \leqslant x_1, x_2, y_1, y_2 \leqslant B$, then it is apparent that $D_k^*(B) = [B] V_k(B)^2$. From Heath-Brown [8] it now follows easily that

$$D_k^*(B) = 4B^5 + O(B^{4+\varepsilon} + B^{3+\mu_k+\varepsilon})$$

where

$$\mu_k = \frac{2}{\sqrt{k}} + \frac{2}{k-1}.$$

Finally, we put

$$\tau_k = \frac{7}{174} - 5k^{-\frac{1}{4}} - \mu_k.$$

**Theorem 2.** *Let $k \geqslant 3$. Then, for each $\varepsilon > 0$, one has*

$$U_k(B) - D(B) \ll B^{4+2\mu_k+\varepsilon}$$

*and*

$$U_k(B) - D_k^*(B) - \Delta(B) \ll B^{4-\tau_k+\varepsilon}.$$

We note that whenever $k \geqslant 24$, one has $2\mu_k < 1$, and so for the latter values of $k$, the conclusion of Theorem 2 is superior to that of Theorem 1. Meanwhile, when $k \geqslant 241\,684\,793$, one finds that $\tau_k > 0$, and one then has a two-term asymptotic formula for $U_k(B)$. Presumably, at least when $k$ is large enough, the linear spaces of solutions responsible for these two main terms provide all of the solutions to the system (2). If that were the case, then on writing $\Delta_0(B)$ for the number of solutions counted by $\Delta(B)$ with $x_5 = y_5$, it would follow that

$$U_k(B) = D(B) + \Delta(B) - \Delta_0(B).$$

It is straightforward to adjust our arguments so as to count all solutions of (2) with $|x_i| \leqslant B$ and $|y_i| \leqslant B$ $(1 \leqslant i \leqslant 5)$. The upper bounds of the revised theorems remain unchanged provided that additional linear spaces of solutions are incorporated into revised definitions of $D(B)$, $D_k^*(B)$ and $\Delta(B)$. The definitions of the latter objects are then rather more complicated, and depend on the parity of the exponent $k$.

Throughout, we use $\varepsilon$ to denote a positive number and apply the convention that, whenever $\varepsilon$ occurs in a statement, then it is asserted that this statement is valid for any value of $\varepsilon > 0$. The implicit constants in Vinogradov's well-known notation, $\ll$ and $\gg$, and in Landau's symbols, will depend at most on $\varepsilon$ and $k$.

## 2. Smaller exponents

Let $k \geqslant 3$. We consider $k$ as fixed once and for all, and will therefore usually suppress dependence on $k$ in our notation. Our argument involves the set $\mathcal{C} = \mathcal{C}_k(B)$ of integers represented by the polynomial $x^k - y^k$, with $1 \leqslant x, y \leqslant B$. We denote by $\mathcal{C}^*$ the set of integers $n \in \mathcal{C}$ that are uniquely represented in the latter form, and we write $\mathcal{E} = \mathcal{C} \setminus \mathcal{C}^*$. We note for future reference that the number of representations of a non-zero integer $n$ in the shape $x^k - y^k$, with $1 \leqslant x, y \leqslant B$, is $O(B^\varepsilon)$.

Now let

$$f(\alpha) = \sum_{1 \leqslant x \leqslant B} e(\alpha x^k)$$

denote the classical Weyl sum, where as usual, we write $e(z)$ for $e^{2\pi i z}$. Our strategy is to estimate the Fourier coefficient

$$c(n) = \int_0^1 |f(\alpha)|^4 e(-\alpha n)\, \mathrm{d}\alpha \tag{3}$$

in mean square as we sum over $n \in \mathcal{C}$. It follows from orthogonality that $c(n)$ counts the number of representations of the integer $n$ in the shape $n = x_1^k + x_2^k - x_3^k - x_4^k$, with $1 \leqslant x_i \leqslant B$ $(1 \leqslant i \leqslant 4)$, and so, on making use of the opening paragraph of this section, it is evident that

$$\sum_{1 \leqslant y < x \leqslant B} c(x^k - y^k)^2 \ll \sum_{n \in \mathcal{C}^*} c(n)^2 + B^\varepsilon \sum_{n \in \mathcal{E}} c(n)^2. \tag{4}$$

**Lemma 1.** *One has*

$$\sum_{n \in \mathcal{C}} c(n)^2 \ll B^5 (\log B)^{\varepsilon - 3}.$$

*Proof.* Suppose first that $k \geqslant 4$. Then, by Bessel's inequality, it follows from (3) that

$$\sum_{n\in\mathcal{C}} c(n)^2 \leqslant \sum_{n\in\mathbb{Z}} |c(n)|^2 \leqslant \int_0^1 |f(\alpha)|^8 \,\mathrm{d}\alpha.$$

But Theorem B of Vaughan [17] shows that

$$\int_0^1 |f(\alpha)|^8 \,\mathrm{d}\alpha \ll B^5(\log B)^{\varepsilon-3},$$

and so the desired conclusion is immediate. It might be worth remarking that this argument makes implicit use of paucity results for sums of two $k$-th powers (see Hooley [11, 12], Skinner and Wooley [13]) required by Vaughan to obtain the sharpest conclusions available from his method.

Turning our attention next to the situation wherein $k = 3$, we apply the Hardy-Littlewood method. Let $\mathfrak{M}$ denote the union of the intervals $|q\alpha - a| \leqslant B^{-9/4}$ with $0 \leqslant \alpha \leqslant 1$, $0 \leqslant a \leqslant q \leqslant B^{3/4}$ and $(a, q) = 1$. Also, put $\mathfrak{m} = [0, 1] \setminus \mathfrak{M}$. For a measurable set $\mathfrak{B} \subset [0, 1]$, let

$$c(n, \mathfrak{B}) = \int_{\mathfrak{B}} |f(\alpha)|^4 e(-\alpha n) \,\mathrm{d}\alpha.$$

An application of Bessel's inequality, followed by reference to Theorem B of Vaughan [16], in the augmented form given by Boklan [1], establishes that

$$\sum_{n\in\mathcal{C}} |c(n, \mathfrak{m})|^2 \leqslant \int_{\mathfrak{m}} |f(\alpha)|^8 \,\mathrm{d}\alpha \ll B^5(\log B)^{\varepsilon-3}.$$

But it follows from the methods of Chapter 4 of Vaughan [18] (see, in particular, section 4.4) that one has

$$\int_{\mathfrak{M}} |f(\alpha)|^4 \,\mathrm{d}\alpha \ll B^{1+\varepsilon},$$

whence the estimate $c(n, \mathfrak{M}) \ll B^{1+\varepsilon}$ holds uniformly in $n$. It follows that

$$\sum_{n\in\mathcal{C}} |c(n, \mathfrak{M})|^2 \ll B^{2+\varepsilon}\mathrm{card}(\mathcal{C}) \ll B^{4+\varepsilon}.$$

But $c(n) = c(n, \mathfrak{M}) + c(n, \mathfrak{m})$, and so, in the case $k = 3$, the estimate claimed in Lemma 1 now follows from the trivial upper bound

$$\sum_{n\in\mathcal{C}} |c(n)|^2 \ll \sum_{n\in\mathcal{C}} |c(n, \mathfrak{M})|^2 + \sum_{n\in\mathcal{C}} |c(n, \mathfrak{m})|^2.$$

□

Before proceeding further, it is useful to establish an auxiliary estimate for the cubic moment of $c(n)$.

**Lemma 2.** *One has*

$$\sum_{n\in\mathbb{Z}} c(n)^3 \ll B^{\frac{13}{2}+\varepsilon}.$$

*Proof.* We begin by observing that from (4) and Lemma 1, one has the upper bound

$$\sum_{1\leqslant x,y\leqslant B} c(x^k-y^k)^2 \ll B^{5+\varepsilon}; \tag{5}$$

here the elementary bound $c(0)\ll B^{2+\varepsilon}$ is sufficient to reintroduce the terms with $x=y$ into (5). Now write

$$I_1=\int_0^1\int_0^1 |f(\alpha)^4 f(\alpha+\beta)^4 f(\beta)^2|\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{6}$$

Then on considering the underlying diophantine equations, it follows from (5) that

$$I_1 \ll B^{5+\varepsilon}. \tag{7}$$

Next we apply the Weyl differencing lemma (see, for example, Lemma 2.3 of Vaughan [18]) to show that

$$|f(\beta)|^4 \leqslant (2B+1)\sum_{|h_1|\leqslant B}\sum_{|h_2|\leqslant B}\sum_{z\in J(h_1,h_2)} e\big(\beta h_1h_2p(z;h_1,h_2)\big), \tag{8}$$

where $J(h_1,h_2)$ is a suitable subinterval of $[1,B]$, and $p(z;h_1,h_2)$ is the polynomial

$$p(z;h_1,h_2)=(h_1h_2)^{-1}\big((z+h_1+h_2)^k-(z+h_1)^k-(z+h_2)^k+z^k\big).$$

By Hua's Lemma (see Lemma 2.5 of Vaughan [18]), one finds that $h_1h_2p(z;h_1,h_2)$ vanishes for at most $O(B^{2+\varepsilon})$ triples $(z,h_1,h_2)$ with $1\leqslant z\leqslant B$ and $|h_i|\leqslant B$ $(i=1,2)$. With these observations in mind, we define

$$F(\beta)=\sum_{h_1,h_2,z} e\big(\beta h_1h_2p(z;h_1,h_2)\big),$$

where the sum is restricted to triples $(h_1,h_2,z)$ with $1\leqslant z\leqslant B$, $|h_i|\leqslant B$ $(i=1,2)$ and $h_1h_2p(z;h_1,h_2)\neq 0$.

Consider the integral

$$I_2=\int_0^1\int_0^1 |f(\alpha)^4 f(\alpha+\beta)^4 f(\beta)^6|\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{9}$$

Then, on isolating terms in (8) for which $h_1h_2p(z;h_1,h_2)=0$, it follows from (6) that

$$I_2 \ll B^{3+\varepsilon}I_1+BI_3, \tag{10}$$

where

$$I_3=\int_0^1\int_0^1 |f(\alpha)^4 f(\alpha+\beta)^4 f(\beta)^2|F(-\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

By orthogonality, the mean value $I_3$ is equal to the number of integral solutions of the system

$$x_1^k+x_2^k-x_3^k-x_4^k=x_5^k+x_6^k-x_7^k-x_8^k=x_9^k-x_{10}^k+h_1h_2p(z;h_1,h_2),$$

with $1\leqslant x_i\leqslant B$ $(1\leqslant i\leqslant 10)$, $1\leqslant z\leqslant B$, $|h_j|\leqslant B$ $(j=1,2)$, subject to $h_1h_2p(z;h_1,h_2)\neq 0$. It follows from Hua's Lemma (Lemma 2.5 of Vaughan [18]) that the number of available choices for $(x_1,\dots,x_8)$ satisfying these equations is bounded above by

$$\int_0^1 |f(\alpha)|^8\,\mathrm{d}\alpha \ll B^{5+\varepsilon}.$$

Fixing any such choice of $(x_1,\dots,x_8)$, and also any of the $O(B^2)$ available choices for $(x_9,x_{10})$, it follows that $h_1,h_2$ and $p(z;h_1,h_2)$ are divisors of the fixed non-zero integer $x_5^k+x_6^k-x_7^k-x_8^k-x_9^k+x_{10}^k$. Hence, there are at most $O(B^\varepsilon)$ possible choices for $h_1$, $h_2$ and $p(z;h_1,h_2)$. But given $h_1$, $h_2$ and $p(z;h_1,h_2)$, the number of possible choices for $z$ is $O(1)$. It follows that

$$I_3 \ll (B^{5+\varepsilon})(B^2)(B^\varepsilon) \ll B^{7+2\varepsilon},$$

and from (7) and (10) we conclude thus far that

$$I_2 \ll B^{8+\varepsilon}. \tag{11}$$

To complete the proof of Lemma 2, we now observe that one has

$$\sum_{n\in\mathbb{Z}} c(n)^3 = \int_0^1\int_0^1 |f(\alpha)f(\alpha+\beta)f(\beta)|^4\,\mathrm{d}\alpha\,\mathrm{d}\beta,$$

as is readily confirmed by considering the underlying diophantine system. On applying Schwarz's inequality, and recalling (6) and (9), it therefore follows from (7) and (11) that

$$\sum_{n\in\mathbb{Z}} c(n)^3 \leqslant (I_1I_2)^{\frac12} \ll B^{\frac{13}{2}+\varepsilon},$$

as required. □

We are now equipped to embark on the proof of Theorem 1. For each $k\geqslant 3$, there is a positive number $\delta=\delta_k$ for which $\mathrm{card}(\mathcal{E})=O(B^{2-\delta})$. In order to confirm this claim, we recall that $\mathcal{E}$ is the subset of the integers $n$ with $|n|\leqslant B^k$, having more than one representation in the form $n=x^k-y^k$, with $1\leqslant x,y\leqslant B$. The work of Hooley [10, 11, 12] therefore establishes that $\delta_k=\frac13-\varepsilon$ is a permissible choice for this exponent, and indeed even larger permissible values of $\delta_k$ may be found in the work of Heath-Brown [8] and Browning [3]. An application of Hölder's inequality therefore leads from Lemma 2 to the upper bound

$$\sum_{n\in\mathcal{E}} c(n)^2 \leqslant \Big(\sum_{n\in\mathbb{Z}} c(n)^3\Big)^{\frac23}\Big(\sum_{n\in\mathcal{E}} 1\Big)^{\frac13} \ll B^{5-\frac{\delta}{4}}.$$

Moreover, the work of Hooley [10, 11, 12] also shows that $c(0)=2B^2+O(B^{2-\delta})$. On substituting these estimates into (4), and recalling Lemma 1, we therefore conclude that

$$\sum_{1\leqslant x,y\leqslant B} c(x^k-y^k)^2-[B]c(0)^2 \ll B^5(\log B)^{\varepsilon-3}+B^{5-\frac{\delta}{4}+\varepsilon},$$

whence

$$\sum_{1\leqslant x,y\leqslant B} c(x^k-y^k)^2-4B^5 \ll B^5(\log B)^{\varepsilon-3}.$$

On interpreting the left hand side of this inequality in terms of the underlying diophantine equations, the conclusion of Theorem 1 follows at once.

## 3. Larger exponents

In this section we briefly consider the consequences of recent work of Heath-Brown for the problem at hand. Let $r(n)$ denote the number of representations of the integer $n$ in the shape $n = x_1^k + x_2^k - x_3^k$, with $1 \leqslant x_i \leqslant B$ $(1 \leqslant i \leqslant 3)$, and satisfying the condition that $x_1 \neq x_3$ and $x_2 \neq x_3$. As we have observed before, the number of representations of the non-zero integer $n$ in the form $n = x^k - y^k$, with $1 \leqslant x, y \leqslant B$, is $O(B^\varepsilon)$. Hence, it follows from the diophantine interpretation of $c(n)$, as defined in (3), that whenever $n \neq 0$, one has

$$c(n) = \sum_{1 \leqslant z \leqslant B} r(n + z^k) + O(B^{1+\varepsilon}).$$

But it follows from the argument of the proof of Theorem 13 of Heath-Brown [8] that $r(n) \ll B^{\mu_k + \varepsilon}$, where $\mu_k$ is defined as in the preamble to Theorem 2, and therefore

$$c(n) \ll B^{1+\mu_k+\varepsilon}. \tag{12}$$

We apply this estimate in two distinct ways. First, from (4), we have

$$\begin{aligned} \sum_{1 \leqslant x,y \leqslant B} c(x^k - y^k)^2 - [B]c(0)^2 &\ll B^\varepsilon \sum_{1 \leqslant y < x \leqslant B} (B^{1+\mu_k+\varepsilon})^2 \\ &\ll B^{4+2\mu_k+3\varepsilon}. \end{aligned} \tag{13}$$

But Heath-Brown [8] has shown that

$$c(0) - 2B^2 \ll B^{1+\varepsilon} + B^{\mu_k+\varepsilon},$$

and thus the first conclusion of Theorem 2 follows by considering the diophantine system underlying the left hand side (13).

In a second direction, we observe that the work of Heath-Brown [9] shows that the number of integral solutions of the equation

$$x_1^k + x_2^k - y_1^k - y_2^k = x_5^k - y_5^k,$$

with $1 \leqslant x_i, y_i \leqslant B$ $(i = 1, 2, 5)$ and $\{x_1, x_2, y_5\} \neq \{y_1, y_2, x_5\}$, is at most $O(B^{3-\sigma_k+\varepsilon})$, where $\sigma_k = \frac{7}{174} - 5k^{-\frac{1}{4}}$. The same bound obviously applies when the indices 1 and 2 are replaced by 3 and 4. Hence, returning to (2) and isolating the solutions there in which $x_5 = y_5$, we deduce that

$$\sum_{1 \leqslant x,y \leqslant B} c(x^k - y^k)^2 - D_k^*(B) - \Delta(B) \ll B^{3-\sigma_k+\varepsilon} \max_{1 \leqslant y < x \leqslant B} c(x^k - y^k).$$

By (12), the right hand side here does not exceed $O(B^{4+\mu_k-\sigma_k+\varepsilon})$. This completes the proof of Theorem 2.

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